A Class of Exact Solutions to the Blandford-Znajek Process

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Abstract

We analyze the constraint equation giving allowed solutions describing fields and currents in a force-free magnetosphere around a rotating black hole. Utilizing the divergence properties of the energy and angular-momentum fluxes for physically allowed solutions, we conclude that poloidal surfaces are independent of the radial coordinate for large values of r. Using this fact and the Znajek regularity condition, we explicitly derive all possible exact solutions admitted by the constraint equation for r independent poloidal surfaces, which are given in terms of the electromagnetic angular velocity function $\Omega = 1/a\sin^2\theta$, where a is the angular momentum per unit mass of the black hole.

Blandford and Znajek [1] proposed a mechanism whereby the rotational energy of black holes could be extracted through electromagnetic processes. In this model, the black-hole magnetosphere is force-free, and the currents and fields are determined self-consistently in the Kerr geometry. They derived a constraint equation for the functions governing the system, and then imposed a regularity condition [2] at the black-hole event horizon.

The general relativistic approach of Blandford and Znajek [1] was recast in terms of a 3+1 absolute space and global time formalism by Thorne and collaborators [3, 4, 5] and, more recently, by Komissarov [6]. Using the approach of Ref. [6], we [7] rederived the Blandford and Znajek constraint equation in the $3+1$ formalism, from which far-field solutions were then derived that matched the Znajek regularity condition. Here we derive the first exact class of solutions for the constraint equation.

Our absolute space is described by a surface of constant time t in the Boyer-Lindquist coordinates (t, r, θ, φ) of the Kerr geometry, with a metric of the form

$$ds^2 = (\beta^2 - \alpha^2)dt^2 + 2\beta_\varphi d\varphi dt + \gamma_{rr}dr^2 + \gamma_{\theta\theta}d\theta^2 + \gamma_{\varphi\varphi}d\varphi^2 \tag{1}$$

(see Refs.[6, 7] for the values of the metric coefficients and a fuller discussion about the derivation of some the results used here).

In a force-free situation, the electric field E is transverse to the magnetic field B, so that $E \cdot B = 0$. Stationarity and axisymmetry imply that the toroidal component of the electric field, $E_\varphi = 0$. Consequently there exists a vector $\omega = \Omega\partial_\varphi$ such that $E = -\,\omega \times B$. Here Ω is the angular velocity of the electromagnetic field, and surfaces of constant Ω define poloidal surfaces. The constitutive equations relating the electromagnetic field and its dual for materials with zero dielectric and magnetic susceptibilities are

$$E = \alpha D + \beta \times B \ , \text{ and } H = \alpha B - \beta \times D \ . \tag{2}$$

Because B has zero divergence, the poloidal (r, θ) component of B can be written as

$$B_P = \frac{\Lambda}{\sqrt{\gamma}}(-\Omega_{,\theta}\partial_r + \Omega_{,r}\partial_\theta) \ . \tag{3}$$

Here $\gamma = \det(\gamma_{ij})$, and Λ is an arbitrary function that is constant on poloidal surfaces (such functions are called poloidal functions).

The electric charge density ρ is given by

$$\sqrt{\gamma}\rho = \partial_r[\frac{\Lambda}{\alpha\sqrt{\gamma}}(\gamma_{\varphi\varphi}\Omega + \beta_\varphi)\gamma_{\theta\theta}\Omega_{,r}] + \partial_\theta[\frac{\Lambda}{\alpha\sqrt{\gamma}}(\gamma_{\varphi\varphi}\Omega + \beta_\varphi)\gamma_{rr}\Omega_{,\theta}]. \tag{4}$$

The toroidal component J^φ of the electric current density vector is given by

$$\sqrt{\gamma}J^\varphi = \partial_r[\frac{\Lambda}{\alpha\sqrt{\gamma}}(\alpha^2 - \beta^2 - \beta_\varphi\Omega)\gamma_{\theta\theta}\Omega_{,r}] + \partial_\theta[\frac{\Lambda}{\alpha\sqrt{\gamma}}(\alpha^2 - \beta^2 - \beta_\varphi\Omega)\gamma_{rr}\Omega_{,\theta}]. \qquad (5)$$

As can be seen, the quantities ρ and J^φ are uniquely described by the poloidal functions Ω and Λ and the metric coefficients. The vanishing of the curl of E means that $B\cdot\nabla H_\varphi = 0$, and therefore, that H_φ is poloidal and therefore a function of Ω alone. We write the constraint equation as

$$\frac{1}{2\Lambda}\frac{dH_\varphi^2}{d\Omega} = \alpha(\rho\Omega\gamma_{\varphi\varphi} - J_\varphi). \qquad (6)$$

The physical meaning of eq. (6) is that when Ω and Λ are correctly picked, the right hand side of the above equation is a function of Ω alone, so that it is possible to integrate and obtain an expression for H_φ. It is important to realize that when Ω, Λ, and H_φ are fixed, the fields and currents are uniquely prescribed.

Given the fields and currents, the flux of energy is given by the Poynting vector $S = E\times H$, which is divergence-free for a force-free, time-independent system. The radial Poynting flux $S^r = -\Omega H_\varphi B^r$. Likewise, the angular-momentum flux vector is divergence-free, and the radial angular-momentum flux $L^r = -H_\varphi B^r$. The net rates of energy and angular-momentum extraction from a rotating black hole are given, respectively, by

$$\frac{d\mathcal{E}}{dt} = \int S^r\sqrt{\gamma_{rr}}dA = -\int H_\varphi\Omega B^r\sqrt{\gamma_{rr}}dA\ , \qquad (7)$$

and

$$\frac{d\mathcal{L}}{dt} = \int L^r\sqrt{\gamma_{rr}}dA = -\int H_\varphi B^r\sqrt{\gamma_{rr}}dA\ . \qquad (8)$$

Since the above two expressions only differ by the presence of Ω in the right hand side, as in [7], we conclude that Ω is asymptotically r independent. With this in mind, for the remainder of this paper, we will assume that $\Omega_{,r} = 0$, as an initial step of research into the search for exact solutions. Consequently, Ω is a function of θ alone, and poloidal surfaces are surfaces of constant θ. Inserting Eqs. (4) and (5) into Eq. (6) for our r-independent Ω, the constraint equation reduces to the form

$$\frac{1}{2\Lambda}\frac{dH_\varphi^2}{d\Omega} = \frac{\alpha\gamma_{\varphi\varphi}}{\sqrt{\gamma}}[\Omega\partial_\theta(\frac{\Lambda}{\alpha\sqrt{\gamma}}(\gamma_{\varphi\varphi}\Omega + \beta_\varphi)\gamma_{rr}\Omega_{,\theta}) + \partial_\theta(\frac{\Lambda}{\alpha\sqrt{\gamma}}(\beta^2 - \alpha^2 + \beta_\varphi\Omega)\gamma_{rr}\Omega_{,\theta})]\ . \qquad (9)$$

For a consistent formulation of the theory of axisymmetric, stationary, force-free magnetospheres, the above assumption implies that H_φ and Λ are to be a function of θ alone. For the

case of a Kerr black hole in Boyer-Lindquist coordinates, the constraint equation becomes

$$\frac{1}{2\Lambda}\frac{dH_\varphi^2}{d\Omega} = \frac{\sin\theta}{\rho^2}[\Omega\partial_\theta(\frac{\Lambda\Omega_{,\theta}}{\sin\theta}(\gamma_{\varphi\varphi}\Omega+\beta_\varphi)) + \partial_\theta(\frac{\Lambda\Omega_{,\theta}}{\sin\theta}(\beta^2-\alpha^2+\beta_\varphi\Omega))]\ . \tag{10}$$

Here $\rho^2 = r^2 + a^2\cos^2\theta$. Expanding the above equation to order $(1/r^3)$ [7] gives

$$\begin{aligned}-\frac{1}{2f(\theta)}\frac{dH_\varphi^2}{d\theta} &= -\Omega\sin\theta\frac{d}{d\theta}(f\Omega\sin\theta) + \frac{\sin\theta}{r^2}[-a^2\Omega\sin^2\theta\frac{d}{d\theta}(f\Omega\sin\theta) + \frac{d}{d\theta}(\frac{f}{\sin\theta})] \\ &+ 2M\frac{\sin\theta}{r^3}[a\Omega\frac{d}{d\theta}(f\sin\theta(1-a\Omega\sin^2\theta)) - \frac{d}{d\theta}(\frac{f}{\sin\theta}(1-a\Omega\sin^2\theta))]\ ,\end{aligned} \tag{11}$$

where $f(\theta) \equiv -\Lambda\Omega_{,\,\theta} \equiv A_{\varphi,\,\theta}$. Since poloidal functions are function of θ alone, all the terms proportional to the inverse powers of r must vanish identically for appropriate choices of f and Ω since H_φ is to be a poloidal function. Consequently, only the zeroth-order term survives, implying that

$$H_\varphi^2 = \pm H_0^2 + (f\Omega\sin\theta)^2. \tag{12}$$

The choice of Ω is determined by the Znajek [2] regularity condition, which is given by

$$H_\varphi = \frac{\sin\theta}{\rho_+^2}(2r_+M\Omega - a)f, \tag{13}$$

where the subscript $+$ indicates that the relevant quantities are to be evaluated at the event horizon and $\rho_+^2 = r_+^2 + a^2\cos^2\theta$. From Eqs. (12) and (13), we can eliminate H_φ and find the relation between f and Ω. Explicitly,

$$\pm H_0^2 = \frac{\sin^2\theta}{\rho_+^4}[(4r_+^2M^2 - \rho_+^4)\Omega^2 - 4r_+Ma\Omega + a^2]f^2. \tag{14}$$

Lemma 1 *If $\Omega \neq 1/a\sin^2\theta$, then f is given by the expression*

$$f = \frac{B_0\sin\theta}{\sqrt{\mid (a\Omega\sin^2\theta)^2 - 1\mid}}\ , \tag{15}$$

where B_0 is a constant.

Proof 1. If f is to be a solution to Eq. (9) for a given form of Ω, then the pair (f,Ω) should remove all the r-dependence in Eq. (11). In particular, the vanishing of the $1/r^2$ term implies that

$$a^2\Omega\sin^2\theta\frac{d}{d\theta}(f\Omega\sin\theta) = \frac{d}{d\theta}(\frac{f}{\sin\theta}). \tag{16}$$

As was shown in Ref. [7], Eq. (15) is the unique solution to the above equation when $\Omega \neq 1/a\sin^2\theta$. §

Eqs. (14) and (15) can be used to completely determine all the possible allowable forms of Ω (when $\Omega \neq 1/a\sin^2\theta$). To this end define

$$\Omega_\pm = \frac{a}{2Mr_+ \pm \rho_+^2}, \tag{17}$$

noting that $\Omega_- = 1/a\sin^2\theta$.

Therefore, a necessary condition that the pair (f,Ω) would generate a self consistent solution to the stationary, axis-symmetric, force-free solution for a magnetosphere in the Kerr geometry is that they satisfy Eq. (14) and Eq. (15) for fields to be regular at the event horizon (as long as $\Omega \neq 1/a\sin^2\theta$).

Lemma 2 *If $\Omega \neq 1/a\sin^2\theta$, the only choices for Ω are*

$$\Omega = \tilde{\Omega}_\pm = \frac{\tilde{A}\Omega_+ \pm \tilde{B}\Omega_-}{\tilde{A} \mp \tilde{B}} \tag{18}$$

where $\tilde{A} = B_0^2 \neq 0$ (if $B_0 = 0$, the fields are trivial), and $\tilde{B} = H_0^2\rho_+^4\Omega_+\Omega_-$.

Proof 2. From Eq.(14) we see that

$$f^2 = \frac{\pm H_0^2\rho_+^4}{a}\frac{\Omega_+\Omega_-^2}{(\Omega-\Omega_+)(\Omega-\Omega_-)}. \tag{19}$$

Here the $\pm$ factor is to ensure that $f^2 \geq 0$. Similarly we find from Eq. (15) that

$$f^2 = \frac{B_0^2}{a^2\sin^2\theta \mid (\Omega-\Omega_-)(\Omega+\Omega_-) \mid}. \tag{20}$$

Equating the right-hand sides of the last two equations, we see that

$$B_0^2 \mid \Omega - \Omega_+ \mid = H_0^2\rho_+^4\Omega_+\Omega_- \mid \Omega + \Omega_- \mid . \tag{21}$$

The above equation has the unique solution given by Eq. (18). §

In the event that $H_0 \to 0$, we see that $\tilde{\Omega}_\pm \to \Omega_+$, so that the Ω_- solution is never realized by $\tilde{\Omega}_\pm$ since $B_0 \neq 0$.

Lemma 3 *No solutions exist to the constraint equation that satisfy the Znajek event horizon regularity condition when $\Omega = \tilde{\Omega}_\pm$.*

Proof 3. If $\Omega = \tilde{\Omega}_\pm$ satisfies the constraint equation (to all orders in r), then f is given by Eq. (15). It is then sufficient to show that $\Omega = \tilde{\Omega}_\pm$, along with f as given in Eq. (15) does not satisfy Eq. (11) to order $1/r^3$. Vanishing of the $1/r^3$ term in Eq. (11) implies that

$$\frac{dg^2}{d\theta} \sin\theta (1 - a\tilde{\Omega}_\pm \sin^2\theta) = 2g^2 \cos\theta (a\tilde{\Omega}_\pm \sin^2\theta + 1), \tag{22}$$

where $g = f(1 - a\tilde{\Omega}_\pm \sin^2\theta)$. Inserting the expression for f, and $\tilde{\Omega}_\pm$ in the definition of g, we find that

$$g^2 = \frac{\pm 1}{r_+^2 + a^2} (H_0^2 \rho_+^4 - B_0^2 \rho_+^2 \sin^2\theta)\,, \tag{23}$$

where $\pm$ ensures that $g^2 > 0$. Inserting Eq. (23) in Eq. (22) we find that

$$-\frac{2Mr_+ B_0^2}{a^2} = \sin^4\theta [B_0^2 a^2 \sin^2\theta - B_0^2 \rho_+^2 - 2H_0^2 a^2 \rho_+^2]. \tag{24}$$

The above equation will not be satisfied because the left hand side is independent of θ unlike the right hand side. Therefore, we reach a contradiction. §.

Lemma 4 $\Omega = \Omega_-$ *is an exact solution to the constraint equation (Eq. (10)) where* Λ *is any arbitrary poloidal function.*

Proof 4. To simplify the discussion, we first make the following observations

$$\begin{aligned}
\beta^2 - \alpha^2 + 2\beta_\varphi \Omega_- + \gamma_{\varphi\varphi}\Omega_-^2 &= \frac{\rho^2}{a^2 \sin^2\theta} \\
\beta^2 - \alpha^2 + \beta_\varphi \Omega_- &= -1 \\
\gamma_{\varphi\varphi}\Omega_- + \beta_\varphi &= \frac{r^2 + a^2}{a}\,.
\end{aligned} \tag{25}$$

The constraint equation (Eq. (10)) can be rewritten as

$$\begin{aligned}
\frac{1}{2\Lambda}\frac{dH_\varphi^2}{d\Omega} = \sin\theta \left(\frac{\Lambda\Omega_{-,\theta}}{\sin\theta}\right)_{,\theta} \frac{\beta^2 - \alpha^2 + 2\beta_\varphi\Omega_- + \gamma_{\varphi\varphi}\Omega_-^2}{\rho^2} + \\
\Lambda\Omega_-\Omega_{-,\theta}\frac{(\gamma_{\varphi\varphi}\Omega_- + \beta_\varphi)_{,\theta}}{\rho^2} + \Lambda\Omega_{-,\theta}\frac{(\beta^2 - \alpha^2 + \beta_\varphi\Omega_-)_{,\theta}}{\rho^2} \\
= \Omega_-\left(\frac{\Lambda\Omega_{-,\theta}}{a\sin\theta}\right)_{,\theta}
\end{aligned} \tag{26}$$

The right hand side is clearly a poloidal function, thus removing the only constraint for arbitrary values of Λ. §

It is easily checked that Ω_- satisfies the regularity condition when $H_0 = 0$, and when $H_\varphi = +f\Omega \sin\theta$. Therefore, from the above lemma, we see that

$$H_\varphi = \frac{2}{a^2}\Lambda \frac{\cos\theta}{\sin^4\theta}. \tag{27}$$

If we put all the lemmas together, we get our main result:

Theorem *When poloidal surfaces are surfaces of constant θ, the unique class of solutions to the stationary, axisymmetric force-free magnetosphere that is regular on the event horizon of a Kerr black hole is generated by the function $\Omega = \Omega_-$. The entire degree of freedom in this theory lies in the poloidal but otherwise arbitrary function Λ.*

The importance of this result is that all exact solutions to the Blandford-Znajek process when $\Omega_{,\,r} = 0$ are constructed from the various choices of Λ and Ω_-, with Λ chosen to give physically allowed expressions for fields, charges, and current densities along the poles.

Corollary *It is impossible to extract energy (and angular momentum) from a stationary, axisymmetric force-free magnetosphere (that is regular on the event horizon) of a Kerr black hole when $\Omega_{,\,r} = 0$.*

Proof. From Eq. (3), Eq. (7), Eq. (8) and Eq. (27) we see that

$$\frac{d\mathcal{E}}{dt} = -\frac{8\pi}{a^4}\int_0^\pi \frac{\Lambda^2\cos^2\theta}{\sin^9\theta} d\theta \leq 0\ , \tag{28}$$

and

$$\frac{d\mathcal{L}}{dt} = -\frac{8\pi}{a^3}\int_0^\pi \frac{\Lambda^2\cos^2\theta}{\sin^7\theta} d\theta \leq 0\ .\S \tag{29}$$

The Ω_+ solution of Ref. [7] is an approximate solution that may be realized as the far-field limit of an exact solution only if poloidal functions become r dependent (as we get closer to the event horizon). This solution has the nice feature that it yields positive energy extraction that is in accord with results of numerical simulations [8, 9] that utilize magnetic fields sustained by external accretion disks. As such, the inability to extract energy from a black hole for the set of exact solutions we have derived indicates that the condition $\Omega_{,\,r} = 0$ must necessarily be relaxed.

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